PHYSICAL REVIEW E, VOLUME 65, 015201(R)

From ballistic to Brownian motion through enhanced diffusion in vertex-splitting polygonal and disk-dispersing Sinai billiards

Valery B. Kokshenev and Eduardo Vicentini
Departamento de Física, Universidade Federal de Minas Gerais, ICEx, Caixa Postal 702, CEP 30123-970, Belo Horizonte, Brazil
(Received 27 July 2001; published 19 December 2001)

Boundary-collision orbit statistics in vertex-splitting rational polygonal and disk-scattering billiards is studied using deterministic and stochastic schemes. On increasing the number of vertices in pseudointegrable polygons, the diffusion exponent, deduced from the mean-square orbit displacement, exhibits a crossover from a ballistic to a superdiffusive regime, characteristic of chaotic Sinai billiards.

DOI: 10.1103/PhysRevE.65.015201 PACS number(s): 05.45.Gg, 05.40.Fb, 45.50.Tn

INTRODUCTION

Rational polygonal classical billiards are polygons whose vertex angles are rational multiples of π. They are known [1–5] to be nonchaotic, with a null Lyapunov exponent and Kolmogorov entropy. A delicate problem emerges when rational polygons are employed [6,7] to approximate chaotic billiards by circumscribing a boundary: what happens to the regular motion in the polygon in the ‘‘infinite-vertex’’ limit? On one hand, the approximation associated with ‘‘quantization’’ of motion can be realized to arbitrary precision; on the other hand, the *correspondence principle* does not work for nonchaotic polygonal systems [8]. The problem is closely related to a violation of the conditions of integrability in polygons, in view of the splitting of orbits at the vertices [2,8]. The vertex-splitting effects cause chaoticlike statistical changes in the associated quantum level spectra [6] and even exhibit positive Lyapunov exponents, if these are estimated with a finite precision [7]. In contrast, a recent study of algorithmic orbital complexity suggests [8] that the vertex-angle effects are logarithmically short with time and polygonal billiards, even those considered with finite precision are ‘‘mostly ordered’’ [8]. In this Rapid Communication we report on insights into the controversial issue of vertex-splitting effects in polygons obtained from the orbit-collision statistics developed for integrable and chaotic billiards.

BILLIARD COLLISION STATISTICS

The collision distribution function $D(n,t)$ is introduced with respect to the Liouville measure as a probability for a given classical particle (of unit mass and unit velocity) to undergo n random collisions in a time interval t (see, e.g., Ref. [9]). It is characterized by the mean collision number

$$n_c(t) \equiv \langle n \rangle_c = \int_0^\infty n D(n,t) dn = \frac{t}{\tau_c} \tag{1}$$

and the rms (root-mean-square) deviation $\Delta n_c(t) = \langle (n - n_c)^2 \rangle_c^{1/2}$. Equation (1) may also be treated as a statistical definition of the mean collision time τ_c, if $t \gg \tau_c$ through the observable $n_c(t)$. This is in agreement with studies of scatter-collision [9,10] and wall-collision [11,12] statistics, which deal with the billiard collision time [9,10,13,14] $\tau_c = \pi A/P$, given by the accessible area A and the perimeter P of a billiard table (or a scatterer) in integrable (or dispersing) systems. Regular polygonal billiards bounded by rational polygons of m equal sides, hereafter m-gons, are introduced by circumscribing below a circle of radius R. Their area $A_m = (mR^2/2)\sin 2\pi/m$ and perimeter $P_m = 2mR \sin \pi/m$ provide the mean collision time $\tau_{cm} = (\pi R/2)\cos \pi/m$. In the ‘‘infinite-vertex’’ m-gon limit, i.e., in the ∞-gon, we naturally arrive at the circle-billiard (CB) with mean time $\tau_{cR} = \pi R/2$, equal to $\tau_{c\infty}$, and the mean collision number $n_{cR}(t) = n_{c\infty}(t)$. Similar findings were established for the *average* coding length for the curved-by-circle ∞-gon [see Eq. (6) in Ref. [8]].

DETERMINISTIC APPROACH

In view of the fact that the billiard wall-collision angle φ, counted off the normal to the boundary, is preserved by elastic reflections, the intrinsic and transient dynamics in integrable billiards can be described [12] by the so-called φ-family orbit sets. In the particular case of regular motion in m-gons, where φ is treated as a second constant of motion [8], a φ-orbit set is given by a characteristic time $\tau_m(\varphi)$ that obeys [12,14] $\langle \tau_m(\varphi) \rangle_c = \tau_{cm}$. Additionally, by using the mean frequency equation [see Eq. (5) in Ref. [12]] we introduce the random n_m and the mean n_{cm} collision numbers:

$$n_m(\varphi,t) = \frac{t}{\tau_m(t\varphi)}, \quad n_{cm} = \langle n_m(\varphi,t) \rangle_c = \frac{t}{\tau_{cm}}, \tag{2}$$

defined by reduced angles $\varphi = [0, \varphi_{cm}]$ in the uniformly occupied three-dimensional (3D) phase space Ω_m through the characteristic times

$$\tau_m(\varphi) = (\pi R/2\varphi_{cm}) \cos(\pi/m) \times \sin\varphi_{cm} \begin{cases} \cos^{-1}\varphi, & \text{odd } m, m/2 \\ \cos\varphi_{cm}\cos^{-1}(\varphi - \varphi_{cm}) & \text{even } m/2, \end{cases} \tag{3}$$

with $\varphi_{cm} = \pi/2m$ and $\varphi_{cm} = \pi/m$, respectively, for the *odd-gon* and the *even-gon* cases. These observables are exemplified by the square billiard with $\tau_4(\varphi) = \sqrt{2}R(\sin\varphi + \cos\varphi)^{-1}$ and by the CB with [12] $\tau_R(\varphi) = 2R\cos\varphi$. Equation (3) for $m=4$ is deduced from Eq. (2) through straightforward calculation of the number of collisions $n_4(\varphi,t)$

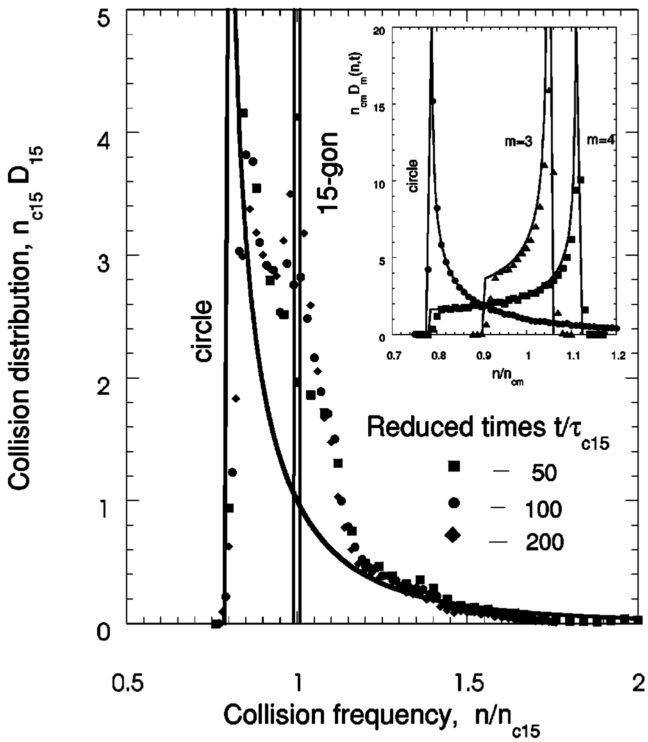

FIG. 1. Wall collision distributions against reduced collision numbers in m-gonal billiards. Reduction is given by Eq. (2) $n_{cm} = t/\tau_{cm}$. Points: numerical simulation data on $n_{c15}D_{15}(n,t)$ for different observation times, $t=50$, 100, and $200\tau_{c15}$. Lines: prediction for $m=15$ by Eq. (5) and that for the CB. Inset: numerical data for the triangle ($m=3$) and the square ($m=4$) billiards compared with Eq. (5), and that for the CB.

$=\varphi_{c4}t(\sin\varphi+\cos\varphi)\tau_{c4}^{-1}$ with the two distinct orthogonal sides in the 4-gon for a given *φ-orbit* from the number of intersections of a *φ-trajectory* in the correspondent [Lorentz gas (LG)] square lattice. Clearly this scheme disregards vertex-splitting effects and, therefore, gives rise to a *regular-motion* part of the statistics introduced in Eq. (1) by the distribution $D_4^{(reg)}=\varphi_{c4}^{-1}|\partial\varphi_4(n,t)/\partial n|$. The latter is found through $\varphi_4=(\frac{1}{2})\arcsin[(n\tau_{c4}/t\varphi_{c4})^2-1]$; given as the inverse function of n_4,

$$D_4^{(reg)}(n,t)=\frac{16}{\pi^2 n_{c4}\sqrt{2}}\sin^{-1}\left[\frac{\pi}{4}-\frac{1}{2}\arcsin\left(\frac{4}{\pi}\frac{n}{n_{c4}}^{2}\right)-1\right],$$
$$\text{for}\quad \pi/4<n/n_{c4}<\pi\sqrt{2}/4,\quad\text{otherwise [15]},\quad D_4^{(reg)}=0. \tag{4}$$

Note that the lower and the upper limiting collision frequencies $\upsilon=n/n_{c4}$ are due to the singular orbits, respectively, known "bouncing balls" (see, e.g., Ref. [12]) given by $\varphi\approx\pi/2,0$ equivalent sets, and to the diagonal vertex-to-vertex φ_{c4} set. Exact solutions similar to Eq. (4) may be found also for $m=3,6$ where the rotational symmetry of a polygon is consistent with the translational symmetry of the corresponding 2D lattice. Meanwhile, in view of the existence of the m-fold rotational axis of symmetry, all sides of a m-gon are statistically equivalent, which allows one to extend the proposed estimation scheme for arbitrary m using the one-side-equivalent model [12]. Hence, the wall-collision distribution that preserves Ω_m space "volume" is

$$D_m^{(reg)}(n,t)=\frac{\sin\varphi_m}{n_{cm}\varphi_{cm}^2}\left[1-\left(\frac{n}{n_{cm}}\frac{\sin\varphi_{cm}}{\varphi_{cm}}\right)^2\right]^{-1/2}$$
$$\text{for}\quad \varphi_{cm}\cot\varphi_{cm}\leqslant n/n_{cm}\leqslant\varphi_{cm}/\sin\varphi_{cm} \tag{5}$$

otherwise $D_m^{(reg)}=0$, where φ_{cm} is described in Eq. (3). One can verify that Eqs. (5) and (4) for $m=4$ are numerically equivalent. Additionally, the reduced rms deviation is [15]

$$\frac{\Delta n_{cm}}{n_{cm}}=\begin{cases}\sqrt{\varphi_{cm}(\varphi_{cm}+\cos\varphi_{cm}\sin\varphi_{cm})/2\sin^2\varphi_{cm}-1}\\ \quad\text{odd}\ \ m,m/2\\ \sqrt{\varphi_{cm}(\varphi_{cm}+\sin\varphi_{cm})/2\sin^2\varphi_{cm}-1}\\ \quad\text{even}\ \ m/2.\end{cases} \tag{6}$$

We see that the ordered motion driven by line segments of the boundary in a finite m-gon is characterized by $n_{cm}\sim\Delta n_{cm}\propto t/\tau_{cm}$. Qualitatively the same can be established for the CB statistics given by $n_{c\infty}D_R=\pi^2/16\nu^4(1-\pi^2/16\nu^2)^{1/2}$ shown in Fig. 1 with $\nu=n/n_{cR}$. The regular-motion predictions for the CB, triangle and square billiards are compared with simulation data [16] in the insert of Fig. 1. Deviations from the deterministic motion are due to weak vertex-splitting effects well pronounced at low and high frequencies. This is not the case when $m\gg1$. As seen from Eq. (6), $\Delta n_{c\infty}=0$ and, hence, $D_\infty^{(reg)}(n,t)=\delta(n-n_{c\infty})$, which means that the ∞-gon dynamics is given solely by the trapped $\varphi_\infty=0$-set, by the way distinct from that in the CB. We observe in Fig. 1 that both trapped sets substantially contribute to the simulated motion in the ∞-gon approximated by the 15-gone. The vertex-singular effects, manifested by nonzero $\Delta n_{c15}^{(\exp)}$, tend to balance the conflicting 15-gon and CB statistics exhibited by the $\varphi\approx0$-set and the $\varphi\approx\pi/2$-set, respectively, below and above the mean frequency. Within this context the question arises as to whether the observed vertex-"ordered" effects are strong enough in the ∞-gon to modify, say, the "sliding" $\varphi\approx\pi/2$-set with long-lived $[\tau_\infty^{(reg)}(\varphi)\gg\tau_{c\infty}]$ orbits into the correspondent "whispering gallery" $\varphi\approx\pi/2$ set of short-lived $[\tau_R(\varphi)\ll\tau_{c\infty}]$ orbits, known in the CB, in order to match the distinct high-frequency tails (see Fig. 1) and, thereby, to restore the correspondence principle that was claimed in Ref. [8].

STOCHASTIC APPROACH

The dispersing *chaotic* Sinai billiard [17] (SB) is a unit cell of the LG lattice model given by noninteracting particles undergoing elastic collisions with scattering disks (of radius R) periodically situated on a square lattice (of side L). Since the pioneering works [18,19] it has been recognized [20,21] that the low-density LG ($R<R_\alpha,R_\alpha=L/2$) exhibits some features of *superdiffusive dynamics*. This was established [20,21] from the mean-square-trajectory displacement

$\langle\Delta^2 r\rangle_c$, through the late-time divergent diffusion "coefficient." Following up-to-date anomalous diffusion theories [22] we represent this finding through the SB *diffusion dynamics exponent* z_R in explicit form (with expected $1<z_R<2$), i.e., $\langle\Delta^2 r\rangle_c \alpha l_{cR}^2 (t/\tilde{\tau}_{cR})^{2/z_R}$, where l_{cR} $(=\tilde{\tau}_{cR})$ is the mean free path between two sequential scatterer collisions. Treating the stochastic motion of particles in the LG within a generalized random-walk scheme on the square lattice one naturally arrives at Eq. (A3), which provides the rms number

$$\Delta\tilde{n}_{cR}(t) \propto (t/\tilde{\tau}_{cR})^{1/z_R} \tag{7}$$

in the SB, and that

$$\Delta\tilde{n}_{cm}(t) \propto m^{1/2}(t/\tilde{\tau}_{cm})^{1/z_m} \quad \text{for } m \gg 1 \tag{8}$$

to estimate vertex-splitting effects in the *m*-gon.

Following to Bleher [20], anomalous diffusion in the SB is due to the existence of the infinite corridors in the correspondent LG lattice, where the particle move boundlessly long without disk scattering. We speculate that similar long-lived trajectories exist in the phase space of *m*-gons with a large number of vertices. They originate from the aforesaid regular-orbit $\varphi \approx \pi/2$ sets, called "sliding" orbits, with quasidivergent characteristic times $\tau_m^{(reg)} \approx \tau_{c\infty} \cos^{-1}\varphi$. More precisely, in even-gons the relevant time emerging in Eq. (8), is $\tilde{\tau}_{cm} = \tau_m^{(reg)}(\pi/2) = \tau_{c\infty} m/\pi$ with the help of Eq. (3). In the case of odd-gons "mesoscopic" analog of the sliding orbits may be introduced by the $\tilde{\varphi}_m = \pi(1/2 - 1/m)$ set through $\tilde{\tau}_{cm} = \tau_m^{(reg)}(\tilde{\varphi}_m) = \tau_{c\infty} m/\pi$. With account of Eq. (8) these result in $\Delta^2\tilde{n}_{cm} \propto m^{1-2/z_m}$. Therefore, three possible scenarios, caused by vertex-splitting effects, are expected in the ∞-gon and given by (i) "vortex" *disorderlike* (if $\Delta^2\tilde{n}_{c\infty}=\infty$), (ii) sliding *orderlike* ($\Delta^2\tilde{n}_{c\infty}=0$), and (iii) *mixed* order-disorder ($\Delta^2\tilde{n}_{c\infty}<\infty$) dynamics. They are distinguished [22] by (i) subdiffusive ($z_\infty>2$), (ii) superdiffusive ($z_\infty<2$), and (ii) Brownian ($z_\infty=2$) motion regimes.

Results of our numerical study of the wall-collision statistics in the dispersive Sinai and "almost integrable" [1] polygonal billiards based on, respectively, Eqs. (7) and (8), are given in Fig. 2. The inset shows a typical temporal behavior of collision fluctuations in *m*-gons that exhibit deviations from the *ballistic dynamical regime* ($z=1$) associated with the ordered motion in *true integrable* billiards. In the SB one distinguishes three superdiffusive regimes with the dynamical exponents $z_R \approx 1.5$ for $0.05L \leqslant R \leqslant R_\beta$ with [20] $R_\beta = \sqrt{2}L/4$ and $1.5<z_R<2.0$ for $R_\beta<R<R_\alpha$, and normal diffusion with $z_R=2$ for the SB with finite horizon ($R \geqslant R_\alpha$) or the diamond billiard [9,10]. The observed Brownian regime, associated with "hard" *chaos* in the SB is, additionally, characterized by a pure Gaussian distribution for scatter-[9] and wall-collision statistics ensured [23] by $\langle(n-\tilde{n}_{cR})^4\rangle_c^{1/2} = \sqrt{3}\Delta^2\tilde{n}_{cR}$. In curved-by-circle polygonal billiards (see the lower scale in Fig. 2) the ballistic ($z_m=1$) and the superdiffusive ($1<z_m<2$) *order-motion* regimes are also revealed. In the ∞-gon vertex-splitting effects are weak ($z_\infty<2$) but, presumably, present ($z_\infty>1$).

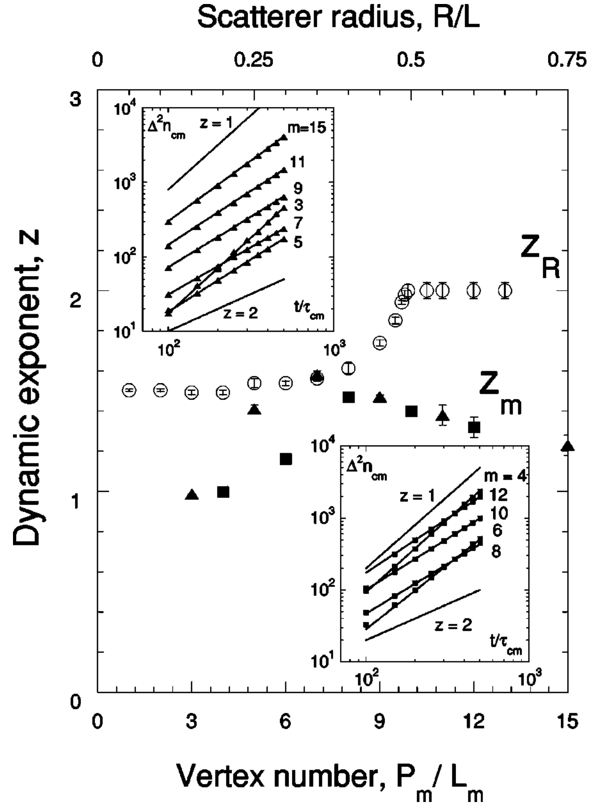

FIG. 2. Diffusion dynamics exponent against scatterer (above) and vertex (below) characteristics in the Sinai and polygonal billiards. Points: open circles—numerical data for the SBs of side $L=1$ for distinct dispersing disks of radius R; closed triangles (or squares)—the data for regular polygons of odd (or even) m equal vertexes, sides of length L_m and perimeters P_m $(=mL_m)$. Insets: points, experimental data for the mean-square collision number deviations on log-log coordinates; lines correspond to Eq. (8).

CONCLUSIONS

We have discussed the interplay between the piece-line regular and vertex-angle singular boundary effects in rational polygons, related by mathematicians to the problem of integrability [1,2,8] and by physicists to the problem of causality and randomness. The proposed deterministic approach to the intrinsic dynamics in *m*-gons reveals weak vertex-singular effects for the cases $m=3,4$. With an increase in the number of vertexes, *irregularlike* motion effects realized by sliding orbits show (see Fig. 1) late-time vertex-memory effects that prevent the establishment of the expected [8] dynamics-motion correspondence in the *geometrical corresponding* system. More specifically, exploration of the correspondence principle borrowed in Ref. [8] from quantum mechanics, where a "quantized" system within the quasiclassical approximation follows the same trajectories as its classical analog, has no justification for classical billiards approximated by vertex-splitting polygons.

We have discussed distinct motion regimes suggested by the stochastic approach. The mixed *order-disorder* regime emerges as chaos in the finite-horizon SB (see Fig. 2 for $z_R=2$). The *disorderlike* regime, common to the chaotic SB, is unstable in *m*-gons ($z_m<2$) and, thus, the "vortex"-like excitations, assumed by preservation of local angular momenta and ensured by the m-fold rotational symmetry, do not survive. Conversely, the *orderlike* dynamics, stabilized by the underlying sliding orbits, exhibits pronounced deviations ($z_m > 1$) from the ballistic-type order-motion regimes and simulates chaotic effects similar to those in the dispersing SB. In conclusion, we have demonstrated how the order-to-chaos crossover in classical systems is driven through the controlled order-disorder-like dynamics regimes.

ACKNOWLEDGMENTS

The authors are grateful to Maria Carolina Nemes and Marcos A. M. de Aguiar for numerous illuminative discussions. Thanks are also due to Mario J. D. Carneiro, Marcos G. E. da Luz, and Alejandro M. F. Rivas for helpful debates and to Ronald Dickman for a critical reading of the manuscript. Financial support of the Brazilian agency CNPq is also acknowledged.

APPENDIX

We deal with the LG model related to the chaotic SB and also associated with the "chaotic" one-equivalent-side model of m-gon with $m \gg 1$. Assuming that the vertex-disorder effects predominate we adopt within the *stochastic approximation* that a given particle intersects the periodic lattice cells (billiard boundaries) after every collision with a scatterer or a vertex. The random number of collisions $n = \Sigma n_i$ is given through counting the number of intersections n_i with an i boundary ($i = 1,2,...,m$) any of m statistically equivalent sides. Thus, we obtain for the mean $n_{cm} = m\langle n_i \rangle_c$ and the fluctuation

$$\Delta^2 n_{cm} = \sum_{i=1}^{m=4} \langle (\Delta n_i)^2 \rangle_c + 2\sum_{i>j} \langle \Delta n_i \Delta n_j \rangle_c \tag{A1}$$

of the random numbers, where $\Delta n_i = n_i - \langle n_i \rangle_c$. Connection with the ($m=4$) *random square displacement* $\Delta^2 r = \Delta^2 x + \Delta^2 y$ [given by $\Delta^2 x = l_{cx}^2 (n_x^+ - n_x^-)^2$ in the x direction] is established through $n = n_x^+ + n_x^- + n_y^+ + n_y^-$. Here n_x^+ (or n_y^-) accounts for intersections of a particle that moves in a positive (or negative) direction of the axis x (or y). Correspondingly, the random square length of a free motion between the two consequent particle-wall collisions, $l_{cx}^2 + l_{cy}^2$, is related to l_{cR} ($= \tilde{\tau}_{cR}$) in the SB, established by $l_{cR}^2 = \langle l_{cx}^2 \rangle_c + \langle l_{cy}^2 \rangle_c = 2\langle l_{cx}^2 \rangle_c$ and estimated in the isotropic approximation. These provide

$$\langle \Delta^2 r \rangle_c = 2\tilde{\tau}_{cR}^2 [\langle n_i^2 \rangle_c - \langle n_i n_j \rangle_c (1 - \delta_{ij})], \tag{A2}$$

where δ_{ij} is the Kronecker symbol. In the stochastic approximation we ignore correlations between reflections from different walls, thus, $\langle \Delta n_i \Delta n_j \rangle_c = \langle \Delta n_i \rangle_c^2 = 0$ and $\langle n_i n_j \rangle_c = \langle n_i \rangle_c^2$ in the last terms in Eqs. (A1) and (A2), respectively. Also, Eq. (A1) is reduced to $\Delta^2 n_{cm} = m(\langle n_i^2 \rangle_c - \langle n_i \rangle_c^2)$ that results in, with help of Eq. (A2), the desirable estimates for wall-collision rms fluctuation

$$\Delta n_c(t) = \sqrt{2}\, \frac{\langle \Delta^2 r \rangle_c^{1/2}}{\tilde{\tau}_{cR}} \quad \text{or} \quad \sqrt{\frac{m}{2}}\, \frac{\langle \Delta^2 r \rangle_c^{1/2}}{\tilde{\tau}_{cm}} \tag{A3}$$

in SBs or in m-gons, respectively.

[1] For review see E. Gutkin, J. Stat. Phys. **83**, 7 (1996).
[2] A. N. Zemlyakov *et al.*, Math. Notes **18**, 760 (1975).
[3] P. J. Richens and M. V. Berry, Physica D **2**, 495 (1981).
[4] B. Echardt, J. Ford, and F. Vivaldi, Physica D **13**, 339 (1984).
[5] E. Gutkin, Physica D **19**, 311 (1986).
[6] T. Cheon and T. D. Cohen, Phys. Rev. Lett. **62**, 2769 (1989).
[7] J. L. Vega, T. Uzer, and J. Ford, Phys. Rev. E **48**, 3414 (1993).
[8] G. Mantica, Phys. Rev. E **61**, 6434 (2000).
[9] P. L. Garrido and G. Gallavotti, J. Stat. Phys. **76**, 549 (1994).
[10] P. L. Garrido, J. Stat. Phys. **88**, 807 (1997).
[11] V. B. Kokshenev and M. C. Nemes, Physica A **275**, 70 (2000).
[12] E. Vicentini and V. B. Kokshenev, Physica A **295**, 391 (2001).
[13] W. Bauer and G. F. Bertsch, Phys. Rev. Lett. **65**, 2213 (1990).
[14] N. Chernov, J. Stat. Phys. **88**, 1 (1997).
[15] The rms deviation $\Delta n_{c4}/n_{c4} = (\varphi_{c4}^2 + \varphi_{c4}/2 - 1)^{1/2}$, calculated within *regular* φ sets in the Ω_4, corresponds to direct estimates by Eqs. (1) and (4) that exposes the ergodicity property [1].
[16] For experimental details see Ref. [12].
[17] Y. G. Sinai, Russ. Math Surveys **25**, 137 (1979).
[18] J. Machta and R. Zwazing, Phys. Rev. Lett. **50**, 1959 (1983).
[19] A. Zacherl *et al.*, Phys. Rev. Lett. **114**, 317 (1986).
[20] P. M. Bleher, J. Stat. Phys. **66**, 315 (1992).
[21] P. Dahlqvist, J. Stat. Phys. **84**, 773 (1996).
[22] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[23] V. B. Kokshenev and E. Vicentini (unpublished).